

ONE-DIMENSIONAL FOLIATIONS ON TOPOLOGICAL MANIFOLDS

SERGIY MAKSYMENKO, EUGENE POLULYAKH

ABSTRACT. Let X be an $(n+1)$ -dimensional manifold, Δ be a one-dimensional foliation on X , and $p: X \rightarrow X/\Delta$ be a quotient map. We will say that a leaf ω of Δ is *special* whenever the space of leaves X/Δ is not Hausdorff at ω . We present necessary and sufficient conditions for the map $p: X \rightarrow X/\Delta$ to be a locally trivial fibration under assumptions that all leaves of Δ are non-compact and the family of all special leaves of Δ is locally finite.

1. INTRODUCTION

Study of the topological structure of flow lines foliations has a long history and leads back to H. Poincaré. The question when a partition into curves is a foliation was considered by H. Whitney [46], [47]. In two-dimensional case one-dimensional foliations appeared as level-sets of pseudo-harmonic functions in W. Kaplan [20], [21].

Let Δ be a one-dimensional foliation on \mathbb{R}^2 , and \mathbb{R}^2/Δ be the space of leaves endowed with the quotient topology. Notice that \mathbb{R}^2/Δ is usually non-Hausdorff. W. Kaplan [20] showed that

- (1) the quotient map $p: \mathbb{R}^2 \rightarrow \mathbb{R}^2/\Delta$ is a locally trivial fibration with fiber \mathbb{R} ;
- (2) there exists at most countably many leaves $\{\omega_i\}_{i \in A}$ of Δ such that the complement $\mathbb{R}^2 \setminus \{\omega_i\}_{i \in A}$ is a disjoint union $\bigsqcup_{j \in B} S_j$, where each S_j is homeomorphic with $(0, 1) \times \mathbb{R}$ so that the lines $t \times \mathbb{R}$, $t \in (0, 1)$, correspond to the leaves of Δ ;
- (3) there exists a pseudoharmonic function (without singularities) $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ whose foliation by connected components of level-sets coincides with Δ .

See also W. Boothby [5], [6], M. Morse and J. Jenkins [16], [17], [18], [19] and M. Morse [33], [32] for extensions of Kaplan's results to foliations with singularities.

A. Haefliger and G. Reeb [13] studied general one-dimensional *non-Hausdorff* manifolds and showed, in particular, that the above result (3) of W. Kaplan can be deduced from Poincaré-Bendixon theorem, see also [12], [37].

Later C. Godbillon and G. Reeb [9] classified locally trivial fibrations over a *non-Hausdorff* letter Y . Though they considered a very special case their methods clarify the general situation.

The question when for an arbitrary k -dimensional foliation Δ on X the quotient map $p: X \rightarrow X/\Delta$ has homotopy lifting properties was considered in C. Godbillon [10], see also G. Meigniez [29] and [30] for the criterion when p is a Serre fibration or a locally trivial fibration but mostly in smooth category. J. Harrison [14] studied similar problem concerning geodesic flows without compact orbits. Also foliations by flow lines on 3-manifolds are classified by S. Matsumoto [27].

In recent years a progress in the theory of Hamiltonian dynamical systems of small degrees of freedom increased an interest to the structure of level-sets functions on surfaces, see e.g.

2010 *Mathematics Subject Classification.* 57R30, 55R10.

Key words and phrases. foliation, non-compact surface, fiber bundle, selection.

A. Fomenko and A. Bolsinov [4], A. Oshemkov [34], V. Sharko [41], [42], V. Sharko and Yu. Soroka [43], E. Polulyakh and I. Yurchuk [36], E. Polulyakh [35].

Homotopy properties of foliations on surfaces glued from strips similarly to (2) are studied in S. Maksymenko and E. Polulyakh [25] and [26] and Yu. Soroka [44].

In [26] the authors extended Kaplan's result (2) to foliations on arbitrary non-compact surfaces X . Namely, under certain assumptions including (1), i.e. that $p : X \rightarrow X/\Delta$ is a locally trivial fibration, the topological structure of the closures $\overline{S_j}$ of strips S_j was described.

In the present paper we consider an arbitrary one-dimensional foliation Δ with all non-compact leaves on a topological manifold X . Our main result gives necessary and sufficient conditions for the quotient map $p : X \rightarrow X/\Delta$ to be a locally trivial fibration, see Theorem 2.8.

As mentioned above such types of questions were extensively studied. However, the essentially new features of Theorem 2.8 in comparison e.g. with [10], [30] and others, is that we work in C^0 category only and give a characterization in terms of the topology of the quotient space X/Δ .

2. ONE-DIMENSIONAL FOLIATIONS

Let $\mathbb{R}_+^n = \{(x_1, \dots, x_n) \mid x_n \geq 0\}$ be the closed half-space in \mathbb{R}^{n+1} .

Definition 2.1 (cf. [7]). *Let X be an $(n+1)$ -dimensional topological manifold, $n \geq 1$. A **foliated chart** on X of codimension n is a pair (U, φ) , where $U \subset X$ is open and $\varphi : U \rightarrow (a, b) \times B^n$ is a homeomorphism with B^n being an open subset of \mathbb{R}_+^n and $a < b \in \mathbb{R} \cup \{\pm\infty\}$. The set $P_y = \varphi^{-1}((a, b) \times \{y\})$, $y \in B^n$, is called a **plaque** of this foliated chart.*

Definition 2.2 (cf. [7, 45]). *Let $\Delta = \{\omega_\alpha \mid \alpha \in A\}$ be a partition of X into pathconnected subsets ω_α of X . Suppose that X admits an atlas $\{U_i, \varphi_i\}_{i \in \Lambda}$ of foliated charts of codimension n such that, for each $\alpha \in A$ and each $i \in \Lambda$, every pathcomponent of a set $\omega_\alpha \cap U_i$ is a plaque. Then Δ is said to be a **foliation of X of dimension 1** (and **codimension n**) and $\{U_i, \varphi_i\}_{i \in \Lambda}$ is called a **foliated atlas associated to Δ** . Each ω_α is called a **leaf** of the foliation and the pair (X, Δ) is called a **foliated manifold**.*

Remark 2.3. In [20] one-dimensional foliations on the plane were also called *regular families of curves*.

In what follows we will assume that X is endowed with some 1-dimensional foliation Δ . We will also consider only foliated charts included into some (maximal) foliated atlas associated to Δ .

Let ω be a leaf of Δ . If ω is compact, then it is homeomorphic with the circle. Otherwise, there exists a continuous bijection $\phi : \mathbb{R} \rightarrow \omega$. Moreover, if $\bar{\phi} : \mathbb{R} \rightarrow \omega$ is another continuous bijection, then $\phi^{-1} \circ \bar{\phi} : \mathbb{R} \rightarrow \mathbb{R}$ is a homeomorphism, c.f. [10, Proposition 6].

Recall that a continuous map $f : A \rightarrow B$ is called *proper* whenever for each compact $K \subset B$ its inverse image $f^{-1}(K)$ is compact. The following lemma is easy and we leave it for the reader.

Lemma 2.4. *Consider the following conditions on $\omega \in \Delta$:*

- (m) *there exists an embedding $\phi : \mathbb{R} \rightarrow X$ with $\phi(\mathbb{R}) = \omega$;*
- (p) *there exists a proper injective continuous map $\phi : \mathbb{R} \rightarrow X$ with $\phi(\mathbb{R}) = \omega$;*
- (p)' *any injective continuous map $\phi : \mathbb{R} \rightarrow X$ with $\phi(\mathbb{R}) = \omega$ is proper;*
- (c) *ω is a closed subset of X .*

Then the following equivalences hold true:

$$(m) \ \& \ (c) \Leftrightarrow (p) \Leftrightarrow (p').$$

A leaf ω satisfying condition (p) of Lemma 2.4 will be said to be *properly embedded*¹.

The union of all leaves of Δ intersecting a subset $U \subset X$ is called the *saturation* of U and denoted by $\mathbf{S}(U)$. The following lemma is easy to prove.

Lemma 2.5. [11, Proposition 1.5], [45, Theorem 4.10] *If $U \subset X$ is open, then $\mathbf{S}(U)$ is open as well.* \square

If Y is a manifold, then a *trivial* 1-dimensional foliation Δ on the product $\mathbb{R} \times Y$ is a partition of $\mathbb{R} \times Y$ into the lines $\mathbb{R} \times y$, $y \in Y$.

Let Δ_i be a 1-foliation on X_i , $i = 1, 2$. Then an embedding $\psi : X_1 \rightarrow X_2$ will be called *foliated* whenever $\psi(\omega)$ is contained in some leaf of Δ_2 for each leaf $\omega \in \Delta_1$.

In particular, if $\varphi : U \rightarrow (a, b) \times B^n$ is a foliated chart as in Definition 2.1, then its inverse $\psi = \varphi^{-1} : (a, b) \times B^n \rightarrow X$ is an open foliated embedding. In this case the set $P_u = \psi((a, b) \times \{u\})$ is a plaque for each $u \in B^n$.

Space of leaves. Let $Y = X/\Delta$ be the space of leaves and $p : X \rightarrow Y$ be the corresponding quotient map. Endow Y with the *quotient topology* with respect to p . Thus a subset $V \subset Y$ is open if and only if its inverse $p^{-1}(V)$ is open in X .

Notice that for a subset $U \subset X$ its saturation is $\mathbf{S}(U) = p^{-1}(p(U))$. In particular, Lemma 2.5 means that p is an open map.

Evidently, Y is a T_1 -space if and only if each leaf of Δ is a closed subset of X . However, in general, Y is not a Hausdorff space.

Special points. Let $u \in Y$ be a point and β_u be a base of neighborhoods of u . Then the following set

$$\text{hcl}(u) := \bigcap_{V \in \beta_u} \overline{V}$$

will be called the *Hausdorff closure* of u . A point u will be called *special*² if $u \neq \text{hcl}(u)$. Notice that $u \in \text{hcl}(v)$ if and only if any two neighborhoods of u and v intersect. The latter statement is symmetric with respect to u and v , and so it is equivalent to the assumption $v \in \text{hcl}(u)$. However, one easily checks that the property “belong to Hausdorff closure” is not transitive.

Evidently, Y is Hausdorff if and only if $u = \text{hcl}(u)$ for all $u \in Y$, that is when Y has no special points. The set of all special points of Y will be denoted by \mathcal{V} .

We will say that a leaf ω of Δ is *special* if $p(\omega)$ is a special point of Y . In particular, $\Sigma := p^{-1}(\mathcal{V})$ is the set of all special leaves of Δ .

The following lemma gives a characterization of special leaves and extends [9, Proposition 4].

Lemma 2.6. *Let $\omega \in \Delta$ be a leaf and $u = p(\omega)$ be the corresponding point in Y . Then the following conditions equivalent:*

- (1) *u is a special point of Y , and so ω is a special leaf of Δ ;*
- (2) *there exists a point $v \in \text{hcl}(u)$ distinct from u and a sequence $\{w_i\}_{i \in \mathbb{N}}$ converging to both points u and v ;*

¹In the book [45, §16] a leaf is called *proper* if it satisfies condition (m).

²In [13, Definition 3] such a point is called a *branch* point. See also [9].

- (3) *there exist two sequences $\{x_i\}_{i \in \mathbb{N}}$ and $\{y_i\}_{i \in \mathbb{N}}$ in X such that x_i and y_i belong to the same leaf for all $i \in \mathbb{N}$, that is $p(x_i) = p(y_i)$, and*

$$\lim_{i \rightarrow \infty} x_i = x \in \omega, \quad \lim_{i \rightarrow \infty} y_i = y \notin \omega. \quad (2.1)$$

Proof. Equivalence (1) \Leftrightarrow (2) is well known and easy.

(3) \Rightarrow (2). Denote $w_i = p(x_i) = p(y_i)$, $i \in \mathbb{N}$ and $v = p(y)$. Then, by continuity of p , the sequence $\{w_i\}_{i \in \mathbb{N}}$ converges to distinct points u and v . In particular, $v \in \text{hcl}(u)$.

(2) \Rightarrow (3). Choose any points $x \in \omega$ and $y \in \omega' := p^{-1}(v)$, and let $\{U_i\}_{i \in \mathbb{N}}$ and $\{V_i\}_{i \in \mathbb{N}}$ be countable bases of topology on X at x and y respectively. Since p is open, $p(U_i)$ and $p(V_i)$ are open neighborhoods of u and v respectively. But these points are special and $u \in \text{hcl}(v)$, whence $p(U_i) \cap p(V_i) \neq \emptyset$ for all $i \in \mathbb{N}$. Hence there exist $x_i \in U_i$ and $y_i \in V_i$ such that $p(x_i) = p(y_i)$. Then $\{x_i\}_{i \in \mathbb{N}}$ and $\{y_i\}_{i \in \mathbb{N}}$ converge to x and y respectively. \square

Definition 2.7. *Let V be an open subset of \mathbb{R}_+^n , where $n = \dim X - 1$ is the codimension of the foliation Δ . A continuous map $\gamma : V \rightarrow X$ is called a **cross section** of Δ if*

- $\gamma(V \cap \partial \mathbb{R}_+^n) \subset \partial X$;
- *the composition $p \circ \gamma : V \rightarrow Y$ is an injective map, that is $u \neq v \in V$ implies that the images $\gamma(u)$ and $\gamma(v)$ of these points belong to distinct leaves of Δ .*

*If $x \in \gamma(V)$ and ω is a leaf of Δ containing x , then we will also say that γ **passes through** x as well as **through** ω .*

The aim of this paper is to present necessary and sufficient conditions for the map p to be a locally trivial fibration under assumption that all leaves of Δ are non-compact.

Theorem 2.8. *Let X be an $(n + 1)$ -dimensional manifold and Δ be a one-dimensional foliation on X . Suppose that all leaves of Δ are non-compact and the family of all special leaves of Δ is locally finite. Then the following conditions are equivalent.*

- (1) *The quotient map $p : X \rightarrow X/\Delta$ is a locally trivial fibration with fiber \mathbb{R} and Y is locally homeomorphic with \mathbb{R}_+^n (though it is not necessary a Hausdorff space).*
- (2) *For each leaf ω there exists an open saturated neighborhood foliated homeomorphic with $\mathbb{R} \times V$, where V is an open subset of \mathbb{R}_+^n .*
- (3) *For each leaf ω of Δ there exists a cross section passing through ω .*

Remark 2.9. It is proved in [10, Chapter III, Propostion 4 and Corollary] that for arbitrary p -dimensional foliation Δ then the quotient map $p : X \rightarrow X/\Delta$ is a Serre fibration whenever it satisfies a certain variant of homotopy extension property and either has a local section at each point or the quotient X/Δ is a (possibly non-Hausdorff manifold). See also [29] and [30] for extensions.

Our Theorem 2.8 claims that for one-dimensional foliations Δ with locally finite family of special leaves existence of cross sections with open subsets of \mathbb{R}_+^n implies that p is even a locally trivial fibration and X/Δ is a possibly non-Hausdorff manifold.

Remark 2.10. Equivalence between (1) and (2) for $\dim X = 2$ is proved in [26] without assumption that Y is locally homeomorphic with \mathbb{R} . Also in [13, §2.2, Proposition 1] it is show that X/Δ is a 1-manifold for one-dimensional foliation on \mathbb{R}^2 .

Remark 2.11. R. H. Bing [2], [3] constructed a non-manifold $B \subset \mathbb{R}^4$ such that $\mathbb{R} \times B$ is homeomorphic with \mathbb{R}^4 . In other words, \mathbb{R}^4 admits a trivial partition into open arcs (being not a foliation) such that the quotient space B is not a 3-manifold. That example was improved by many authors, see e.g R. Rosen [38], J. Kim [22], J. Bailey [1], L. Rubin [39].

Remark 2.12. E. Dyer and M. Hamstrom [8] studied so called *completely regular* mappings $p : X \rightarrow Y$ between metric spaces such that the inverse images of all points are in a certain sense “uniformly homeomorphic”, and get sufficient conditions when such a map is equivalent to a trivial fibration, see [8, Theorem 7], and also [28], [40] for generalizations. We consider here a similar problem, but now the space Y is not even Hausdorff, and we gave conditions when p is a locally trivial fibration.

The following statement is proved in [43, Theorem 1] for continuous functions $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, and in [30, item 3 at the end of page 3778] for smooth case.

Theorem 2.13. *Let M and N be two manifolds such that $\dim M = \dim N + 1$ and $f : M \rightarrow N$ be a surjective continuous map such that*

- $f(\text{Int } M) = \text{Int } N$ and $f(\partial M) = \partial N$;
- *the partition $\Delta = \{f^{-1}(c) \mid c \in N\}$ of M constitutes a one-dimensional foliation with all non-compact leaves.*

Then f is a locally trivial fibration with fiber \mathbb{R} . In particular, if N is contractible, then f is a trivial fibration.

Proof. We claim that Δ contains no special leaves and each leaf admits a cross section. Then it will follow from Theorem 2.8 that f is a locally trivial fibration with fiber \mathbb{R} .

Absence of special leaves. Let $Y = M/\Delta$ be the space of leaves endowed with the corresponding factor topology. Then f can be written as a composition of the following maps

$$f = \theta \circ p : M \xrightarrow{p} Y \xrightarrow{\theta} N,$$

where θ is the induced continuous bijection. Since N is Hausdorff, it follows that so is Y , and therefore Y contains no special points. Hence Δ contains no special leaves.

Existence of cross sections. Let $x \in M$ and $\varphi : U \rightarrow (-1, 1) \times B^n$ be a foliated chart at x as in Definition 2.2 such that $\varphi(x) = (0, 0) \in (-1, 1) \times B^n$, where $n = \dim N$. Then the map $\gamma : B^n \rightarrow M$ defined by $\gamma(y) = \varphi^{-1}(0, y)$ is a cross section of Δ . \square

In fact, Theorem 2.8 is an easy consequence of the following statements:

Lemma 4.6. *Let ω_0 be a leaf of Δ . Suppose that for each leaf ω of Δ contained in $\mathbf{S}(\overline{\omega_0})$ there exists a cross section γ passing through ω . Then ω_0 is properly embedded.*

Theorem 2.14. *Let $\gamma : V \rightarrow X$ be a cross section intersecting only leaves being simultaneously non-compact, properly embedded, and non-special. Then the saturation $\mathbf{S}(\gamma(V))$ is open and foliated homeomorphic with $\mathbb{R} \times V$.*

The proof of Theorem 2.8 will be given in §3. In §4 we will prove some general preliminary results concerning one-dimensional C^0 foliations being well known for smooth case. In particular we will prove Lemma 4.6. §5 is devoted to the proof of Theorem 2.14 using E. Michael’s theorems about selections of multivalued maps.

3. PROOF OF THEOREM 2.8

(1) \Rightarrow (2), (3). Suppose the quotient map $p : X \rightarrow Y$ is a locally trivial fibration with fiber \mathbb{R} and Y is locally homeomorphic with \mathbb{R}_+^n . This means that for each $\omega \in \Delta$ there exist

- an open neighborhood $V \subset Y$ of its image $u = p(\omega)$ homeomorphic with an open subset of \mathbb{R}_+^n and
- a foliated homeomorphism $\psi : \mathbb{R} \times V \rightarrow p^{-1}(V)$.

Then $p^{-1}(V)$ is an open and saturated neighborhood of ω and ψ is a foliated homeomorphism required by (2).

Moreover, the map $\gamma : V \rightarrow X$ defined by $\gamma(v) = \psi(0, v)$ is a cross section passing through ω . This proves (3).

(3) \Rightarrow (2). Suppose each leaf of Δ admits a local cross section. Then it follows from Lemma 4.6 that all leaves of Δ are properly embedded. Let Σ be a family of all special leaves and $\sigma \in \Sigma$ be a special leaf.

Since each leaf is closed and Σ is a locally finite family, it follows that $\Sigma \setminus \sigma$ is a closed set, whence $X' = (X \setminus \Sigma) \cup \sigma$ is open and saturated and contains no special leaves. Moreover, since each leaf in X' admits a local cross section, it follows from Theorem 2.14 that each leaf $\omega \subset X'$ has an open saturated neighborhood W foliated homeomorphic with $\mathbb{R} \times V$, where V is an open subset of \mathbb{R}_+^n . Then W is also open in X . This proves (2).

(2) \Rightarrow (1). Let $u \in Y$ and $\omega = p^{-1}(u)$ be the corresponding leaf of Δ . Suppose there exist an open $V \subset \mathbb{R}_+^n$ and a foliated homeomorphism $\psi : \mathbb{R} \times V \rightarrow W_\omega$ onto some open and saturated neighborhood W_ω of ω . Since p is an open map, so is the composition $p \circ \psi$. Hence $U_u := p(\psi(W_\omega))$ is an open neighborhood of u in Y . Moreover, the restriction $p \circ \psi|_{0 \times \mathbb{R}^n} : 0 \times V \rightarrow U_u$ is a continuous and open bijection, and so it is a homeomorphism. Thus Y is locally homeomorphic with \mathbb{R}_+^n and the map $p \circ \psi : \mathbb{R} \times V \rightarrow U_u$ is a trivialization of p over U_u , so p is a locally trivial fibration with fiber \mathbb{R} . Theorem 2.8 is completed.

4. PRELIMINARIES

In this section we will assume that X is an $(n + 1)$ -dimensional manifold with $\partial X = \emptyset$ and Δ is a one-dimensional foliation on X .

Some statements in this section are well known for C^1 foliations e.g. [30], and some of them are proved for C^0 case but for the foliations on \mathbb{R}^2 , see e.g. W. Kaplan [20]. However we did not find good exposition in the literature for general C^0 foliations needed in our case and therefore short proofs will be presented. This will also make the paper self-contained.

It will be convenient to regard the graph of a function $f : X \rightarrow \mathbb{R}$ as the following subset

$$\Gamma_f := \{(f(x), x) \mid x \in X\}$$

of $\mathbb{R} \times X$. Thus we switch the coordinates.

Lemma 4.1. *Let Z be a topological space, $f_1, \dots, f_k : Z \rightarrow (a, b)$ be continuous functions such that $f_i(z) < f_j(z)$ for all $i < j$ and $z \in Z$, and*

$$\Gamma_i = \{(f_i(z), z) \mid z \in Z\} \subset [a, b] \times Z$$

be the graph of f_i . Let also $c_1 < c_2 < \dots < c_k \in (a, b)$ be any increasing k -tuple of numbers. Then there exists a self-homeomorphism h of $[a, b] \times Z$ such that

- (a) *h is fixed on $a \times Z$ and $b \times Z$;*
- (b) *$h([a, b] \times z) = [a, b] \times z$ for all $z \in Z$;*
- (c) *$h(\Gamma_i) = c_i \times Z$;*
- (d) *if $f_i(z) = c_i$ for some $z \in Z$ and all $i = 1, \dots, k$, then h is fixed on $[a, b] \times z$.*

Proof. The proof follows from [24, Lemma 6.1.1], see also [26, Lemma 5.2.1]. Let us just mention that the situation can be reduced to the case $[a, b] = [0, 1]$, and that for $k = 1$ the

desired self-homeomorphism h of $[0, 1] \times Z$ can be defined e.g. by

$$h(s, z) = \begin{cases} (s, z), & s \in \{0, 1\}, \\ (s^{\log_{f_1(z)} c_1}, z), & s \in (0, 1). \end{cases}$$

We leave the details for the reader, see Figure 4.1. \square

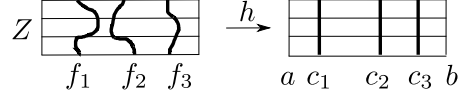


FIGURE 4.1.

Lemma 4.2. *Let W be an open neighborhood of 0 in \mathbb{R}^n and $\gamma : W \rightarrow (a, b) \times \mathbb{R}^n$ be a cross section of the trivial one-dimensional foliation such that $\gamma(0) \in (a, b) \times 0$. Then for each $c \in (a, b)$ there exists an open embedding $\psi : (a, b) \times W \subset (a, b) \times \mathbb{R}^n$ such that*

- (i) $\psi((a, b) \times x) = (a, b) \times \gamma(x)$ for all $x \in W$;
- (ii) $\psi(c, x) = \gamma(x)$ for all $x \in W$, i.e. $\psi^{-1}(\gamma(W)) = c \times W$;
- (iii) if $\gamma(0) = (c, 0)$, then $\psi(t, 0) = (t, 0)$ for all $t \in (a, b)$.

Proof. Let $\pi : (a, b) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the standard projection. Then the assumption that γ is a cross section means that the composition

$$\pi \circ \gamma : W \xrightarrow{\gamma} (a, b) \times \mathbb{R}^n \xrightarrow{\pi} \mathbb{R}^n$$

is an injective map between open subsets of \mathbb{R}^n . Hence by Brouwer's theorem on domain invariance, e.g. [15], $\pi \circ \gamma(W)$ is an open neighborhood of 0 in \mathbb{R}^n . Therefore we get an open embedding

$$\psi : (a, b) \times W \rightarrow (a, b) \times \mathbb{R}, \quad \psi(t, x) = (t, \pi \circ \gamma(x))$$

satisfying (i) and (iii). Then $\psi^{-1}(\gamma(W)) \subset (a, b) \times W$ can be regarded as a graph of certain continuous function $W \rightarrow (a, b)$. Hence we get from Lemma 4.1 that ψ can be composed with a foliated homeomorphism of $(a, b) \times W$ to satisfy (ii), that is to make $\psi^{-1}(\gamma(W))$ being the graph of the constant function $W \rightarrow c$. Moreover, statements (b) and (d) of Lemma 4.1 allow to preserve properties (i) and (iii) respectively. \square

Lemma 4.3. *Let ω be a leaf of Δ , $J_1, J_2 \subset \omega$ be two compact segments such that $J_1 \cap J_2$ is a point, V be an open n -disk, $\varepsilon > 0$, and*

$$\psi_1 : (a - \varepsilon, b + \varepsilon) \times V \rightarrow X, \quad \psi_2 : (b - \varepsilon, c + \varepsilon) \times V \rightarrow X$$

be two open foliated embeddings such that

$$\psi_1([a, b] \times 0) = J_1, \quad \psi_2([b, c] \times 0) = J_2, \quad \psi_1(b, 0) = \psi_2(b, 0) = J_1 \cap J_2,$$

and the union of the images of ψ_1 and ψ_2 does not contain compact leaves of Δ . Then there exists an open neighborhood W of 0 in V and an open foliated embedding

$$\psi : (a - \varepsilon, c + \varepsilon) \times W \rightarrow X$$

such that $\psi([a, c] \times 0) = J_1 \cup J_2$, see Figure 4.2.

Proof. Notice that the assumption that the union of the images of ψ_1 and ψ_2 does not contain compact leaves of Δ implies that for any $u, v \in V$ the union of the arcs

$$\psi_1((a - \varepsilon, b + \varepsilon) \times u), \quad \psi_2((b - \varepsilon, c + \varepsilon) \times v)$$

does not contain a non-trivial loop, so the intersection of these arcs is connected (though possibly empty).

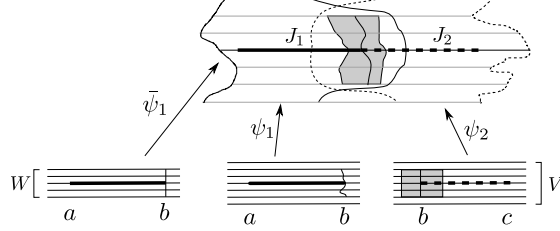


FIGURE 4.2.

Since ψ_1 and ψ_2 are open embeddings, there exist $\delta > 0$ and a small open neighborhood W of 0 in V such that

$$\psi_2((b - \delta, b + \delta) \times W) \subset \text{image}(\psi_1).$$

Then we have an embedding $\gamma : W \rightarrow V$ defined by $\gamma(u) = \psi_1^{-1}(\psi_2(b, u))$, $u \in W$. Hence by Lemma 4.2 one can find an open foliated embedding

$$\bar{\psi}_1 : (a - \varepsilon, b + \varepsilon) \times W \rightarrow X$$

such that

- $\bar{\psi}_1(t, 0) = \psi_1(t, 0)$ for all $t \in (a - \varepsilon, b + \varepsilon)$;
- $\text{image}(\bar{\psi}_1) = \psi_1((a - \varepsilon, b + \varepsilon) \times W)$;
- $\pi \circ \bar{\psi}_1^{-1}(x) = \pi \circ \psi_2^{-1}(x)$ for all $x \in \text{image}(\bar{\psi}_1) \cap \text{image}(\psi_2)$;
- $\bar{\psi}_1(b, u) = \psi_2(b, u)$ for all $u \in W$.

Now define the map $\psi : (a - \varepsilon, c + \varepsilon) \times V \rightarrow X$ by

$$\psi(t, u) = \begin{cases} \bar{\psi}_1(t, u), & t \in (a - \varepsilon, b], \\ \psi_2(t, u), & t \in [b, c + \varepsilon). \end{cases}$$

One easily checks that ψ is an open foliated embedding which coincides with ψ_1 on $(a - \varepsilon, b] \times 0$ and with ψ_2 on $[b, c + \varepsilon) \times 0$. In particular, $\psi([a, c] \times 0) = J_1 \cup J_2$. \square

Corollary 4.4. c.f. [30, Lemma 22] *Let B^n be an open n -disk, ω be a leaf of Δ , and $J \subset \omega$ be a compact segment. Then there exists an open foliated embedding $\psi : (0, 3) \times B^n \rightarrow X$ such that $\psi([1, 2] \times 0) = J$.*

Proof. Let us show that there exists an open set W such that $J \subset W$ and W does not contain compact leaves of Δ . Indeed, since ω is either non-compact or is an embedded circle, it follows that $J \neq \omega$. Fix a point $x \in \omega \setminus J$. As X is a regular space, there exist a pair of disjoint open neighborhoods $U_1 \ni x$ and $U_2 \supset J$. Denote $W = U_2 \cap \mathbf{S}(U_1)$. Then by Lemma 2.5 W is open. Moreover, $J \subset \omega = \mathbf{S}(x) \subset \mathbf{S}(U_1)$, so $J \subset W$. Finally, since $W \subset \mathbf{S}(U_1)$, we see that $\mathbf{S}(y) \cap U_1 \neq \emptyset$ for each $y \in W$, that is $\mathbf{S}(y) \not\subset W$. In other words, W does not contain any leaf of Δ . In particular, W can not contain compact leaves.

Notice that J can be covered by finitely many foliated charts contained in W . Lemma 4.3 allows to replace two consecutive foliated charts with one. Hence the proof follows from that lemma by induction on the number of foliated charts covering J . \square

Cross sections. The following two lemmas describe general properties of cross sections.

Lemma 4.5. *Let $\psi : (a, b) \times B^n \rightarrow X$ be an open foliated embedding. Let also $U = \psi((a, b) \times B^n)$, $P_u = \psi((a, b) \times u)$, $u \in B^n$, be a plaque of ψ , and $\gamma : V \rightarrow X$ be a cross section. Then the following statements hold true.*

- (1) *Suppose $P_u \cap \gamma(V) \neq \emptyset$ for each $u \in B^n$. Then then for each $s \in (a, b)$ the restriction map*

$$\psi|_{\{s\} \times B^n} : \{s\} \times B^n \rightarrow X \quad (4.1)$$

is a cross section of Δ .

- (2) *Suppose $\gamma(v) \in P_u$ for some $u \in B^n$ and $v \in V$. Then there exists an open neighborhood V_v of v in V and an open neighborhood W_v of v in B^n such that*

- $\gamma(V_v) \subset \psi((a, b) \times W_v)$;
- $P_w \cap \gamma(V_v) \neq \emptyset$ for each $w \in W_v$.

In particular, the restriction $\psi|_{s \times W_v} : s \times W_v \rightarrow X$ is a cross sections of Δ .

- (3) *For every $x \in \mathbf{S}(\gamma(V))$ there exist an open subset W of \mathbb{R}^n and a cross-section $\psi_x : W \rightarrow X$ such that $x \in \psi_x(W) \subset \mathbf{S}(\gamma(V))$.*

Proof. (1) Suppose $P_u \cap \gamma(V) \neq \emptyset$ for all $u \in B^n$. Since $\gamma(V)$ intersects each leaf of Δ in at most one point, it follows that distinct plaques P_u and P_v for $u \neq v \in B^n$ belong to distinct leaves of Δ . As B^n is an open subset of \mathbb{R}^n , the map (4.1) is a cross section for each $s \in (a, b)$.

- (2) Consider the following map:

$$\xi = \pi \circ \psi \circ \gamma|_{\gamma^{-1}(U)} : \gamma^{-1}(U) \xrightarrow{\gamma} U \xrightarrow{\psi} (a, b) \times B^n \xrightarrow{\pi} B^n, \quad (4.2)$$

where π is the standard projection to the second coordinate.

Then the assumption $\gamma(v) \in P_u$ for some $u \in B^n$ and $v \in V$ implies that $v \in \gamma^{-1}(U)$ and $\xi(v) = u$.

Since the images of distinct points of V under γ are contained in distinct leaves of Δ , they also belong to distinct plaques of ψ , whence ξ is an injective continuous map between open subsets of \mathbb{R}^n . Hence, by Brouwer theorem on domain invariance ξ is an open map, [23]. In particular, ξ yields a homeomorphism of some open neighborhood V_v of v onto some open neighborhood W_u of u in B^n . This implies that $\gamma(V_v) \subset \psi((a, b) \times W_u)$ and $P_w \cap \gamma(V_v) \neq \emptyset$ for each $w \in W_u$.

- (3) Let ω be the leaf containing x and $y = \gamma(v) = \omega \cap \gamma(V)$. If $x = y$, then one can put $W_x = V$ and $\gamma_x = \gamma$.

Therefore suppose $x \neq y$. Let $J \subset \omega$ be a closed segment with ends x and y . Then by Corollary 4.4 there exists an open foliated embedding $\psi : (0, 3) \times B^n \rightarrow X$ such that $\psi([1, 2] \times 0) = J$, $\psi(1, 0) = x$ and $\psi(2, 0) = y$.

Thus $y = \gamma(v) = \psi(2, 0) \in P_0 = \psi((0, 3) \times 0)$, and so by (2) there exists a neighborhood W of 0 in B^n such that the map $\psi_x : W \rightarrow X$ defined by $\psi_x(w) = \psi(1, w)$ is a cross section with $\psi_x(W) \subset \mathbf{S}(\gamma(V))$. It remains to note that $\psi_x(0) = \psi(1, 0) = x$. \square

Lemma 4.6. *Let ω_0 be a leaf of Δ . Suppose that for each leaf ω of Δ contained in $\mathbf{S}(\overline{\omega_0})$ there exists a cross section γ passing through ω . Then ω_0 is properly embedded, i.e. it satisfies conditions (m) and (c) of Lemma 2.4.*

Proof. If ω_0 is compact, then it is necessarily properly embedded. Therefore assume that ω_0 is non-compact.

(m) Let $\omega \subset \mathbf{S}(\overline{\omega_0})$ be a leaf of Δ . By (3) of Lemma 4.5 for each $x \in \omega$ there exists an open foliated embedding $\psi : (-1, 1) \times B^n \rightarrow X$ such that $\psi(0, 0) = x$ and different plaques of ψ are contained in different leaves of Δ . In particular, ψ homeomorphically maps $(-1, 1) \times \{0\}$ onto an open neighbourhood of x in ω . This implies that ω is an embedded 1-submanifold of X .

(c) Let $x \in \mathbf{S}(\overline{\omega_0}) \setminus \omega_0$. Then decreasing B^n one can assume that the image of ψ does not intersect ω_0 , whence $x \notin \overline{\omega_0}$. From arbitrariness of $x \in \mathbf{S}(\overline{\omega_0})$ we conclude that ω_0 is closed in X . \square

Parallel cross sections. Let $\gamma : V \rightarrow X$ be a cross section and $W \subset V$ be an open subset. Then a cross section $\delta : W \rightarrow X$ *parametrically agrees* with γ , whenever for each $u \in W$ the points $\delta(u)$ and $\gamma(u)$ belong to the same leaf. Also δ is *parallel to* γ if it parametrically agrees with γ and $\delta(W) \cap \gamma(W) = \emptyset$.

Let $\gamma_0, \gamma_1 : V \rightarrow X$ be two parallel cross sections intersecting only non-compact leaves. For each $u \in V$ let ω_u be the leaf containing $\gamma_0(u)$ and $\gamma_1(u)$, $I_u \subset \omega_u$ be the compact segment with ends $\gamma_0(u)$ and $\gamma_1(u)$, and $\text{Int } I_u$ be the interior of I_u . In this situation we will put:

$$L(\gamma_0, \gamma_1) := \bigcup_{u \in V} \text{Int } I_u, \quad K(\gamma_0, \gamma_1) := \bigcup_{u \in V} I_u. \quad (4.3)$$

Lemma 4.7. *There exists a homeomorphism $\psi : [0, 1] \times V \rightarrow K(\gamma_0, \gamma_1)$ such that*

$$\psi([0, 1] \times u) = I_u, \quad \psi(0, u) = \gamma_0(u), \quad \psi(1, u) = \gamma_1(u)$$

for every $u \in V$, see Figure 4.3. In particular, $\psi((0, 1) \times V) = L(\gamma_0, \gamma_1)$. Moreover, $L(\gamma_0, \gamma_1)$ is open in X .

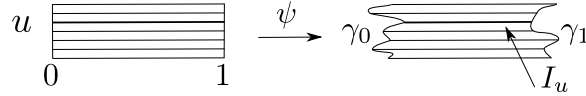


FIGURE 4.3.

Proof. Fix some $\varepsilon > 0$ and denote $J = (-\varepsilon, 1 + \varepsilon)$. Then it follows from Corollary 4.4 and Lemma 4.2 that for each $u \in V$ there exists a neighborhood W_u in V and an open foliated embedding $\psi_u : J \times W_u \rightarrow X$ having the following properties:

- (a) $\psi_u([0, 1] \times u) = I_u$, $\psi_u(0, u) = \gamma_0(u)$, and $\psi_u(1, u) = \gamma_1(u)$;
- (b) $\psi_u(J \times v)$, $\gamma_0(v)$, and therefore $\gamma_1(v)$, are contained in the same leaf of Δ for each $v \in W_u$;
- (c) $\gamma_0(W_u) = 0 \times W_u$ and $\gamma_1(W_u) = 1 \times W_u$.

In particular, this implies that the set

$$L(\gamma_0, \gamma_1) = \bigcup_{x \in V} \psi_u((0, 1) \times W_u)$$

is open in X .

As V is paracompact, there is a locally finite cover $\{W_i\}_{i \in \Lambda}$ of V and for each $i \in \Lambda$ an open foliated embedding $\psi_i : J \times W_i \rightarrow X$ such that $\psi_i([0, 1] \times u) = I_u$ for all $u \in W_i$. Denote $U_i = \psi_i(J \times W_i)$ and $U = \bigcup_{i \in \Lambda} U_i$. Then U is an open neighborhood of $K(\gamma_0, \gamma_1)$ and $\{U_i\}_{i \in \Lambda}$ is a locally finite cover of U .

Let $\{\lambda_i : V \rightarrow [0, 1]\}_{i \in \Lambda}$ be a partition of unity subordinated to the cover $\{W_i\}_{i \in \Lambda}$. Thus $\text{supp}(\lambda_i) \subset W_i$ and $\sum_{i \in \Lambda} \lambda_i(u) = 1$. Let also $p_i : J \times W_i \rightarrow J$ and $q_i : J \times W_i \rightarrow W_i$ be the standard projections, and

$$\mu_i = \lambda_i \circ q_i \circ \psi_i^{-1} : U_i \xrightarrow{\psi_i^{-1}} J \times W_i \xrightarrow{q_i} W_i \xrightarrow{\lambda_i} [0, 1].$$

Then $\text{supp}(\mu_i) = J \times \text{supp}(\lambda_i)$, whence μ_i extends by zero to a continuous function on all of U .

Let $f : U \rightarrow J$ be the function defined by the following rule:

$$f(x) = \sum_{x \in U_i} \mu_i(x) \cdot p_i \circ \psi_i^{-1}(x).$$

Since for each $u \in W_i$ the function $p_i \circ \psi_i^{-1} : I_u \rightarrow [0, 1]$ is homeomorphism which maps $\gamma_0(u)$ and $\gamma_1(u)$ to 0 and 1 respectively, and $\sum_{j \in \Lambda} \mu_j \equiv 1$, we see that the restriction $f|_{I_u}$ is a convex linear combination of orientation preserving homeomorphisms. Therefore $f|_{I_u} : I_u \rightarrow [0, 1]$ is a homeomorphism as well.

Let also $g : U \rightarrow V$ be the map defined by $g(x) = q_i(x)$ whenever $x \in U_i$. Due to (b) this definition does not depend on a particular U_i containing x . Hence g is a well-defined continuous map.

Then the mapping

$$\phi = (f, g) : K(\gamma_0, \gamma_1) \rightarrow [0, 1] \times V$$

is a continuous bijection being also a local homeomorphism, and so it is a homeomorphism. Moreover, $\phi(I_u) = [0, 1] \times u$ for all $u \in V$. Therefore $\psi = \phi^{-1}$ is the required homeomorphism. \square

Lemma 4.8. *Let $\gamma_i : V \rightarrow X$, $i \in \mathbb{Z}$, be a family of pairwise parallel cross sections intersecting only non-compact leaves and $U = \mathbf{S}(\gamma_i(V))$ be the common saturation of their images. Suppose also that the following two conditions hold:*

- (1) $L(\gamma_i, \gamma_{i+1}) \cap L(\gamma_j, \gamma_{j+1}) = \emptyset$ for $i \neq j$;
- (2) $\bigcup_{i=-\infty}^{\infty} K(\gamma_i, \gamma_{i+1}) = U$.

Then U is open in X and foliated homeomorphic with $\mathbb{R} \times V$.

Proof. By Lemma 4.7 for each $i \in \mathbb{Z}$ there exists a homeomorphism

$$\psi_i : [i, i+1] \times V \rightarrow K(\gamma_i, \gamma_{i+1})$$

such that for each $u \in V$

- $\psi_i([i, i+1] \times u)$ is a segment of the leaf of Δ between the points $\gamma_i(u)$ and $\gamma_{i+1}(u)$;
- $\psi_{i-1}(i, u) = \psi_i(i, u) = \gamma_i(u)$.

Therefore we have a homeomorphism

$$\psi : \mathbb{R} \times V \longrightarrow \bigcup_{i=-\infty}^{\infty} K(\gamma_i, \gamma_{i+1}) = U$$

defined by $\psi(t, u) = \psi_i(t, u)$ whenever $t \in [i, i+1]$ and $u \in V$. Moreover, $U = \bigcup_{i=1}^{\infty} L(\gamma_{-i}, \gamma_i)$ is open in X . \square

5. PROOF OF THEOREM 2.14

Let $\gamma : V \rightarrow X$ be a cross section intersecting only leaves being simultaneously non-compact, properly embedded, and non-special. We have to prove that its saturation $\mathbf{S}(\gamma(V))$ is open and foliated homeomorphic with $\mathbb{R} \times V$.

First we will assume that $\partial X = \emptyset$. The proof of the case $\partial X \neq \emptyset$ will follow from the case $\partial X = \emptyset$ by passing to the double $2X$ of X and considering the one-dimensional foliation on $2X$ induced by Δ . It will be given at the end of this section.

Our proof is based on the following statement which will be proved below.

Proposition 5.1. *Let $K \subset X$ be a compact subset. Then one can find two parallel cross sections $\alpha, \beta : V \rightarrow X$ parametrically agreeing with γ and satisfying*

$$\mathbf{S}(\gamma(V)) \cap K \subset L(\alpha, \beta).$$

Moreover, if $\mathbf{A}, \mathbf{B} : V \rightarrow X$ are two parallel cross sections parametrically agreeing with γ , then one can assume that

$$(\mathbf{S}(\gamma(V)) \cap K) \cup K(\mathbf{A}, \mathbf{B}) \subset L(\alpha, \beta).$$

Before proving Theorem 2.14 let us deduce it from Proposition 5.1.

Fix any increasing sequence $K_1 \subset K_2 \subset \dots$ of compact subsets of X such that $X = \bigcup_{i \in \mathbb{N}} K_i$.

Using Proposition 5.1 one constructs a family of parallel cross sections $\alpha_i, \beta_i : V \rightarrow X$, $i \in \mathbb{N}$, parametrically agreeing with γ and such that

- (1) $\mathbf{S}(\gamma(V)) \cap K_i \subset L(\alpha_i, \beta_i)$;
- (2) $K(\alpha_{i-1}, \beta_{i-1}) \subset L(\alpha_i, \beta_i)$ for all $i \geq 2$.

Hence

$$\mathbf{S}(\gamma(V)) = \bigcup_{i \in \mathbb{N}} \mathbf{S}(\gamma(V)) \cap K_i = \bigcup_{i \in \mathbb{N}} L(\alpha_i, \beta_i).$$

Exchanging α_i and β_i if necessary and re-denoting them as follows: $\gamma_{-i} = \alpha_i$, and $\gamma_{i-1} = \beta_i$ for $i \in \mathbb{N}$, one can assume that the sequence of cross sections $\{\gamma_i\}_{i \in \mathbb{Z}}$ satisfies assumptions of Lemma 4.8, see Figure 5.1. Hence $\mathbf{S}(\gamma(V))$ is open and foliated homeomorphic with $\mathbb{R} \times V$. This proves Theorem 2.14 modulo Proposition 5.1.

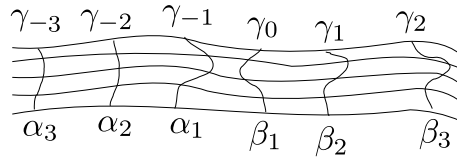


FIGURE 5.1.

The following lemma guarantees existence of local cross sections in Proposition 5.1.

Lemma 5.2. *Let $K \subset X$ be a compact subset. Then for each $u \in V$ one can find an open neighborhood W in V and two parallel cross sections $\alpha, \beta : W \rightarrow X$ parametrically agreeing with γ and such that*

$$\mathbf{S}(\gamma(W)) \cap K \subset L(\alpha, \beta).$$

Proof. Suppose that lemma fails, so there exists $u \in V$ belonging to some leaf ω such that

- for any decreasing sequence W_i of neighborhoods of u in V with $\bigcap_{i \in \mathbb{N}} W_i = \{u\}$

- and any family of pairs of parallel cross sections $\alpha_i, \beta_i : W_i \rightarrow X, i \in \mathbb{N}$, parametrically agreeing with γ

the set

$$\mathbf{S}(\gamma(W_i)) \setminus L(\alpha_i, \beta_i)$$

contains some point $x_i \in K$.

Denote

$$U = \bigcup_{i \in \mathbb{N}} L(\alpha_i, \beta_i).$$

Then one can assume, in addition, that the following properties hold:

- (a) the sequence $\{x_i\}_{i \in \mathbb{N}}$ converges to some point $x \in K$;
- (b) $\omega \subset U$;
- (c) $x_i \notin U$ for all $i \in \mathbb{N}$, whence $x \notin U$ as well, and so $x \notin \omega$.

Indeed, (a) follows from compactness of K .

To prove (b) fix any continuous bijection $\phi : \mathbb{R} \rightarrow \omega$. By assumption ϕ is proper, so one can find $A > 0$ such that $\omega \cap K \subset \phi(-A, A)$. Choose α_i and β_i so that $\alpha_i(W_i) \cap K = \beta_i(W_i) \cap K = \emptyset$,

$$\begin{aligned} \cdots < \alpha_{i+1}(u) < \alpha_i(u) < \cdots < \alpha_1(u) < -A < \\ & < A < \beta_1(u) < \cdots < \beta_i(u) < \beta_{i+1}(u) < \cdots \end{aligned}$$

$\lim_{i \rightarrow +\infty} \alpha_i(u) = -\infty$, and $\lim_{i \rightarrow +\infty} \beta_i(u) = +\infty$. Then we will have that $\omega \subset U$.

Finally, to satisfy (c) choose W_{i+1} so small that $x_i \notin \mathbf{S}(\gamma(W_{i+1}))$ for all $i \in \mathbb{N}$.

Now let ω_i be the leaf of Δ containing x_i , and $y_i = \omega_i \cap \gamma(W_1)$. Then the sequence $\{y_i\}_{i \in \mathbb{N}}$ converges to $y = \gamma(u) \in \omega$. Hence $p(x) \neq p(y) = p(\omega)$, while $p(x_i) = p(y_i) = p(\omega_i)$ for all $i \in \mathbb{N}$. Therefore by Lemma 2.6 ω is a special leaf which contradicts to the assumption. \square

The rest of the proof of Theorem 2.14 is based on E. Michael's result about selections, [31].

Let 2^X be the set of all subsets of X and $\mathcal{E}(X) \subset 2^X$ be the set of all closed subsets of X . Let also $A \subset V$ be a subset and $q : V \rightrightarrows X$ be a multivalued map, i.e. a map $q : V \rightarrow 2^X$. Then a *selection* for the restriction $q|_A$ is a continuous map $\phi : A \rightarrow X$ such that $\phi(x) \in q(x)$ for all $x \in A$.

A multivalued map $q : V \rightrightarrows X$ is called *lower semi-continuous* whenever for each open $U \subset X$ the set

$$T_U := \{x \in V \mid q(x) \cap U \neq \emptyset\}$$

is open in V .

A family $\mathcal{Z} \subset 2^X$ is called *equi- LC^k* , $k \geq 0$, if for every $P \in \mathcal{Z}$, $x \in P$, and a neighborhood U_x of x in X , there exists a neighborhood O_x of x in X such that for every $Q \in \mathcal{Z}$ every continuous map $f : S^m \rightarrow Q \cap O_x$ of an m -sphere ($m \leq k$) is homotopic to a constant map in $Q \cap U_x$.

A topological space Z is called *C^k* , or *k -connected*, $k \geq 0$, if every continuous map $f : S^m \rightarrow Z$ of an m -sphere ($m \leq k$) is homotopic to a constant map.

Theorem 5.3. [31, Theorem 1.2] *Let V be a separable metric space, $A \subset V$ be a closed subset with $\dim(V \setminus A) \leq k + 1$, X a complete metric space, $\mathcal{Z} \subset \mathcal{E}(X)$ be equi- LC^k and $q : V \rightarrow \mathcal{Z}$ be a lower semi-continuous map. Then every selection for $q|_A$ can be extended to a selection for $q|_U$ for some open $U \supset A$. If also every $S \subset \mathcal{Z}$ is C^k , then one can take $U = X$.*

We will use the following particular case of Theorem 5.3.

Corollary 5.4. *Let V be a separable metric space, $\dim V = n$, X be a complete metric space, and $\mathcal{Z} \subset \mathcal{E}(X)$ be an equi- LC^{n+1} family such that each $Q \in \mathcal{Z}$ is contractible. Then every lower semi-continuous multivalued map $q : V \rightarrow \mathcal{Z}$ has a continuous selection.*

Proof of Proposition 5.1 Since V is paracompact, it follows from Lemma 5.2 that there exist

- a locally finite open cover $\mathcal{W} = \{W_i\}_{i \in \mathbb{N}}$ of V with compact closures $\overline{W_i}$, and
- a family of pairs of parallel cross sections $\alpha_i, \beta_i : W_i \rightarrow X$, $i \in \mathbb{N}$, parametrically agreeing with γ

such that

$$\mathbf{S}(\gamma(W_i)) \cap K_i \subset L(\alpha_i, \beta_i),$$

where $K_i = K \cup \mathbf{A}(\overline{W_i}) \cup \mathbf{B}(\overline{W_i})$ whenever the cross sections $\mathbf{A}, \mathbf{B} : V \rightarrow X$ are given, and $K_i = K$ otherwise.

Then it follows from Lemmas 4.7 and 4.1 that for each $i \in \mathbb{N}$ one can find an embedding $\psi_i : [-1, 1] \times W_i \rightarrow X$ such that for each $u \in V$

- (1) $\psi_i([-1, 1] \times u)$ is contained in the leaf of Δ ;
- (2) $\psi_i(-1, u) = \alpha_i(u)$, $\psi_i(0, u) = \gamma(u)$, $\psi_i(1, u) = \beta_i(u)$;
- (3) $\mathbf{S}(\gamma(W_i)) \cap K_i \subset \psi_i((-1, 1) \times W_i)$;
- (4) $\alpha_i(W_i)$ are contained in the same path component of $\mathbf{S}(\gamma(V)) \setminus \gamma(V)$ for all $i \in \mathbb{N}$.

Let $u \in V$, ω_u be the leaf of Δ containing $\gamma(u)$, and $\phi_u : \mathbb{R} \rightarrow \omega_u$ be any bijection satisfying $\phi_u^{-1}(\alpha_i(u)) < 0$, and $\phi_u(0) = \gamma(u)$. Therefore $\phi_u^{-1}(\beta_i(x)) > 0$ for all i such that $u \in W_i$. Then there are two numbers a_u, b_u such that

$$\omega_u \setminus \bigcup_{i : u \in W_i} L(\alpha_i, \beta_i).$$

consists of two half closed intervals $A_u = \phi_u(-\infty, a_u]$ and $B_u = \phi_u[b_u, +\infty)$.

Since ω_u is a properly embedded leaf, it follows that A_u and B_u are closed in X . Moreover, by (3) they do not intersect K .

Define the following two maps $a, b : V \rightarrow \mathcal{E}(X)$, i.e. multivalued mappings $a, b : V \rightrightarrows X$ with closed images, by

$$a(u) = A_u, \quad b(u) = B_u$$

for $u \in V$.

Lemma 5.5. (i) *The maps a and b are lower semi-continuous.*

(ii) *The families $\mathcal{A} = \{A_u \mid u \in V\}$ and $\mathcal{B} = \{B_u \mid u \in V\}$ are equi- LC^k for all $k \geq 0$.*

Proof. It suffices to check (i) and (ii) for a only.

(i) We should check that for each open $U \subset X$ the set

$$T_U = \{u \in V \mid a(u) \cap U \neq \emptyset\}$$

is open as well.

Let $u \in V$ be such that $A_u \cap U \neq \emptyset$, and $x \in A_u \cap U$. Since U is open, one can assume that x is not the end of A_u , that is $\phi_u^{-1}(y) < a_u$.

By assumption $u \in W_i$ for some $i \in \mathbb{N}$. Then by Corollary 4.4 for the closed interval on ω_u between x and $\phi_u(a_u)$ there exists an open neighborhood O of u in W_i and an open foliated embedding $\psi : (-1, 2) \times O \rightarrow X$ such that

- (a) $\psi((-1, 2) \times O) \subset U$;
- (b) $\psi(0, u) = x$;
- (c) $\psi((-1, 2) \times v) \subset \omega_v$;
- (d) $\psi(1, v) = \alpha_i(v)$.

It follows from (a) and (d) that $\psi((-1, 1] \times v) \subset A_v \cap U$, whence $O \subset T_U$. Thus T_U is open, and so a is a lower semi-continuous multivalued map.

(ii) Notice that for each $x \in X$ there exists an open neighborhood U_x such that the intersection of U_x with each leaf ω is either empty or homeomorphic to an open interval. Therefore intersection of U_x with each set A_u is either empty or homeomorphic to $(0, 1)$ or to $(0, 1]$. In the latter two cases $U_x \cap A_u$ is contractible. Hence every continuous map $S^k \rightarrow U_x \cap A_u$ is null homotopic and one can put $O_y = U_y$. This means that \mathcal{A} is equi- LC^k for all $k \geq 0$. \square

Since for each $u \in V$ the sets A_u and B_u are contractible, it follows from Lemma 5.5 that a and b satisfy assumptions of Corollary 5.4. Hence they admit continuous selections $\alpha, \beta : V \rightarrow X$ and these selections are the required cross sections. This completes Proposition 5.1.

Proof of Theorem 2.14. Case $\partial X \neq \emptyset$. We need the following simple lemma whose proof we leave for the reader.

Lemma 5.6. *Let $\xi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the involution defined by*

$$\xi(x_1, \dots, x_{n-1}, x_n) = (x_1, \dots, x_{n-1}, -x_n).$$

*Then for each subset $V \subset \mathbb{R}_+^n$ open in the induced topology of \mathbb{R}_+^n , its **double** $\widehat{V} = V \cup \xi(V)$ is open in \mathbb{R}^n .* \square

Now let

$$\widehat{X} = X_1 \sqcup_{\text{id}:\partial X_1 \rightarrow \partial X_2} X_2$$

be the double of X , i.e. the union of two copies X_1 and X_2 of X glued along their boundaries by the identity map. Let also $\sigma : \widehat{X} \rightarrow \widehat{X}$ be the involution interchanging X_1 and X_2 by the identity map.

Then the foliation Δ on each of the copies of X gives a one-dimensional foliation $\widehat{\Delta}$ on \widehat{X} . Moreover, let \widehat{V} be the double of V as in Lemma 5.6. Then \widehat{V} is open in \mathbb{R}^n and the cross section γ naturally extends to the cross section $\widehat{\gamma} : \widehat{V} \rightarrow \widehat{X}$ of $\widehat{\Delta}$ such that $\widehat{\gamma}|_V = \gamma$ and $\sigma \circ \widehat{\gamma} = \widehat{\gamma} \circ \xi$.

Since $\partial \widehat{X} = \emptyset$, it follows from the boundary-less case of Theorem 2.14 that the saturation $\mathbf{S}(\widehat{\gamma}(\widehat{V}))$ is open in \widehat{X} and foliated homeomorphic with $\mathbb{R} \times \widehat{V}$. That homeomorphism induces a homeomorphism of the open subset

$$\mathbf{S}(\gamma(V)) = \mathbf{S}(\widehat{\gamma}(\widehat{V})) \cap X_1$$

of X_1 onto $\mathbb{R} \times V$. Theorem 2.14 is completed.

6. ACKNOWLEDGMENTS

The authors are sincerely grateful to Olena Karlova for pointing out to Katetov-Tong theorem about characterization of normal spaces which leads us to the proof of Theorem 2.14 using E. Michael selection theorem.

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